Mentored Undergraduate Summer Experience Proposal

Summer 2009

Faculty member:
Andrew Clifford, Associate Professor, Department of Mathematics and Statistics

Years at TCNJ: 13

1 Student Collaborator:
Matthew Davis

Project title:
The Structure of Rank Two Subgroups of Free Groups

(no IACUC or IRB required)

Project and learning plan

Intellectual Merit

A *group* is an abstract number system with one operation, which we call multiplication. Beyond their inherent interest, groups are used in scientific disciplines to describe symmetries of structures and so have broad application. An important class of groups is the collection of *free groups*. These groups are foundational to symmetries of objects which can be given a hyperbolic geometric structure in some sense. Moreover, free groups are the building blocks of all groups in that each group is realized as a factor group of a free group.

A subset of a free group is called a *basis* if every element of the free group can be uniquely expressed as the product of elements from that subset. It is known that every basis of a given free group has the same number of elements. This number is called the rank of the free group. If a subset does not generate the entire free group, it generates a proper *subgroup*. Algorithms have been established which can effectively calculate various properties of subgroups of free groups, including their rank and whether they are malnormal. However, a method to determine whether or not two subgroups are equivalent under an ambient isomorphism has not been determined. Partial results in this direction are well-known. For example, two subgroups that have different ranks cannot be equivalent in this way.

The broad question which this project will address is whether it is possible to determine whether two subgroups of rank 2 are equivalent. In previous work done with an undergraduate collaborator, we have identified geometric descriptions of rank 2 subgroups of a free group and classified these descriptions into three categories, *barbell subgroups*, *theta subgroups*, and *figure eight subgroups*. We have noticed that there are examples from each of these categories which are equivalent. However, we constructed a class of barbell subgroups which are only equivalent to other barbell subgroups and not equivalent to any theta subgroups or figure eight subgroups. Our project will determine whether there are similar examples in the other subgroup classes. It may also be possible to determine an algorithm which determines whether a subgroup has this restrictive property.

Role of Student and Mentor:

As described above, this work will be a continuation of a successful experience I have had with another student. I plan on mirroring the structure of that experience. To that end, I imagine that the first week will be dedicated to ensuring that Matt has the basic exposure to graph theory and group theory that will enable him to generate examples for study. This will take the form of an intensive directed reading in graph theory and a few lectures in free groups to supplement his coursework in Abstract Algebra.

With this background, we will be able to dedicate the following five or six weeks to our project proper. This will entail Matt constructing examples of various rank two subgroups and applying automorphisms of the ambient free group and calculating the resultant subgroup. Predictably, my involvement in this part of the project will entail intensive

interactions with Matt and his calculations. Hopefully, he will develop the skills necessary to allow my role to be more supervisory. At this point, I expect that my role will primarily entail the verification of his calculations and guidance as to where he should be looking for examples for his subsequent calculations. In the middle of this period, I expect that we will be in a position to make conjectures which may be supported by future calculation. Of course, the eventual goal will be the construction of mathematical proofs of some of these conjectures.

Concurrent to these activities, Matt will be either backfilling or expanding his content knowledge with directed readings of texts and journal articles. This will be less intensive than the first week of content development, however. I expect to play a similar, if more relaxed, role during that stage of the project.

Broader Impacts

I will begin by discussing the impact this will have on Matt, who came to The College declaring his intention to pursue a career in academia. Beyond the well-established formative impact such deep immersive experiences have been shown to have on students, his participation in the MUSE program will be an invaluable exposure to the joys and frustrations of his career choice.

Personally, I am also looking forward to discovering the impact that this experience will have on the nature of my teaching. My attitude toward undergraduate research is in the midst of an evolution – from doubting it was possible to successfully accomplish it in pure mathematics through wondering whether is appreciably better than other deep learning experiences towards a better understanding of how these experiences can provide deep satisfaction to both the mentor and mentee. Having found ways to make bring aspects of my research program into several types of independent studies during the academic year, I feel that my next step is to participate in the MUSE program.

Lastly, I would like to presume to comment on the impact our participation will have on my colleagues. I know that several of my colleagues in pure math are at different points in an evolution similar to mine. Moreover, as a department, we have cautiously moved toward developing a more public aspect to our scholarship. In the past, rarely did we share the successes and frustrations of our work with each other. Interestingly, it is through our students that we have come out of our respective research closets. As they became better able and more eager to do high caliber work with the faculty, their participation in the Celebration of Student Achievement, their presentations at regional conferences and their development of a student seminar brought aspects of our research programs into the light. I feel that if we have a successful experience, Matt and I will contribute to this new and exciting trend. Hopefully, in subsequent summers, there will be a critical mass of mathematicians seizing the opportunity that the MUSE program provides.

Proposed Budget:

Student stipend……..$2500
Student housing…….$1305
Faculty Stipend……..$1000

Total……..…….$4805

Name: Matthew Davis
Major: Math (MATA)
Year: 2
Expected Graduation Date: Spring 2011
Request On-campus Housing: Yes

With goals of attending graduate school after TCNJ, and ultimately becoming a research mathematician by profession, participation in the MUSE program will allow me to gain experience that will help me achieve these goals. This experience is vital to my further understanding of the mathematical field. It will allow me the opportunity to challenge myself and expand my mind in ways not attained by simply attending classes. Moreover, doing research on a topic would force me to think about my own new ideas instead of learning someone else's from a book. In addition to contributing to my career goals, partaking in MUSE would significantly influence my scholastic goals. Specifically, I am currently enrolled in Abstract Algebra—a course in which I learn about algebraic groups. However, the research I would be doing in MUSE would analyze groups differently than in Abstract Algebra. This would permit me to indulge in the study of groups in two significantly different ways, giving me entirely new views on the same topic. Consequently, my knowledge and understanding of algebraic groups would grow astronomically. Since the genuine goal of my studies is to become as knowledgeable in pure mathematics as possible, MUSE would help my efforts tremendously.

As stated, I am extremely interested in pure mathematics. My interest in this subject is continuing to grow with every new topic I learn. Because my interest level is so high, I feel that I would excel in the MUSE program. Academically, I have proven that I am capable of achieving great success. Being on the Dean's List every semester I have completed thus far has required much dedication, focus, and commitment to my studies. I will apply the same work ethic to MUSE and expect similar success. Although the above qualifies me to participate in MUSE, I believe my most excellent characteristic is my ability to think outside the box. This feature is incredibly important in math because new discoveries require looking at subjects at unusual angles. I exhibit an equally important quality in the way I conduct myself in the classroom; I always actively participate in all of my classes, asking questions about material as well as questioning if an alternate route to an answer is correct. This shows that I am constantly thinking of new ways to do things, and have a forward thinking mind. Again, this is an important trait that will make me a valuable member of the research team.

My specific interest in the study of groups is even more significant than other areas in math. This is mostly because of the theoretical nature of this topic. I am interested in understanding all there is to know about groups and applying my knowledge to make new discoveries in this field. Given the opportunity, I will contribute to MUSE's success as well as MUSE contributing to mine.